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Space-filling Latin hypercube designs for computer experiments

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Abstract In the area of computer simulation, Latin hypercube designs play an important role. In this paper the classes of maximin and Audze-Eglais Latin hypercube designs are considered. Up to now only several two-dimensional designs and a few higher dimensional designs for these classes have been published. Using periodic designs and the Enhanced Stochastic Evolutionary algorithm of Jin et al. (J. Stat. Plan. Inference 134(1):268–687, 2005), we obtain new results which we compare to existing results. We thus construct a database of approximate maximin and Audze-Eglais Latin hypercube designs for up to ten dimensions and for up to 300 design points. All these designs can be downloaded from the website <http://www.spacefillingdesigns.nl>.

Keywords Audze-Eglais · Computer experiment · Enhanced stochastic evolutionary algorithm · Latin hypercube design · Maximin · Non-collapsing · Packing problem · Simulated annealing · Space-filling

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1 Introduction

A k -dimensional Latin hypercube design (LHD) of n points, is a set of n points $x_i = (x_{i1}, x_{i2}, \dots, x_{ik}) \in \{0, \dots, n-1\}^k$ such that for each dimension j all x_{ij} are distinct. In this definition, we assume that our design space is equal to the $[0; n-1]^k$ hypercube. However by scaling, we can use LHDs for any rectangular design space. Alternative definitions of LHDs also occur in the literature. One alternative definition is to divide each axis into n equally sized bins and randomly select points such that each bin contains exactly one point. However, we refer to this technique as Latin hypercube sampling (LHS). In this paper the term ‘LHD’ thus only refers to the first definition.

An LHD is called maximin when the separation distance $\min_{i \neq j} d(x_i, x_j)$ is maximal among all LHDs of given size n , where d is a certain distance measure. In this paper, we concentrate on the Euclidean (or ℓ^2) distance measure, i.e.,

$$d(x_i, x_j) = \sqrt{\sum_{l=1}^k (x_{il} - x_{jl})^2}, \quad (1)$$

since this measure is often the first choice in practice.

Besides maximin LHDs, we also treat Audze-Eglais LHDs. These LHDs minimize the following objective:

$$\sum_{i=1}^n \sum_{j=i+1}^n \frac{1}{d(x_i, x_j)^2}, \quad (2)$$

where $d(x_i, x_j)$ is again the Euclidean distance between points x_i and x_j . By minimizing this objective, we can also obtain LHDs with “evenly spread” points (Bates et al. 2004).

For both classes of LHDs, we aim to construct a database of the best designs known in literature. We do this by generating new designs and comparing them with existing results. These designs are often approximate maximin or Audze-Eglais designs in the sense that optimality of the objective is not guaranteed. The reason for this is that optimization over the total set of LHDs can be very time-consuming for larger values of k and n . Therefore, in order to find good designs, optimization is often done over a certain class of LHDs or heuristics are used which do not guarantee optimality. The periodic LHDs described in this paper are a good example of the first case. Examples of the second case are simulated annealing used by Morris and Mitchell (1995), the permutation genetic algorithm of Bates et al. (2004) and the Enhanced Stochastic Evolutionary (ESE) algorithm of Jin et al. (2005).

The designs which are best according to the comparison in this paper are added to the website <http://www.spacefillingdesigns.nl> where they can be downloaded for free. As far as we know this is the first extensive online catalogue of maximin and Audze-Eglais LHDs, although there are several catalogues for classical design of experiments, see e.g., the WebDOE™ website of Crary (2008). Crary et al. (2000) developed I-OPT™ to generate designs with minimal integrated mean squared error (IMSE). They found that IMSE-optimal designs can have proximate design points, which they call “twin points”; see also Crary (2002).

Our main motivation for investigating this subject is that maximin and Audze-Eglais Latin hypercube designs are very useful in the area of computer simulation. One important area where computer simulation is used a lot is engineering. Engineers are confronted with the task of designing products and processes. Since physical experimentation is often expensive and difficult, computer models are frequently used for simulating physical characteristics. The engineer often needs to optimize the product or process design, i.e., to find the best settings for a number of design parameters that influence the critical quality characteristics of the product or process. A computer simulation run is usually time-consuming and there is a great variety of possible input combinations. For these reasons, meta-models that model the quality characteristics as explicit functions of the design parameters are constructed. Such a meta-model, also called a (global) approximation model or surrogate model, is obtained by simulating a number of design points. Well-known meta-model types are polynomials and Kriging models. Since a meta-model evaluation is much faster than a simulation run, in practice such a meta-model is used, instead of the simulation model, to gain insight into the characteristics of the product or process and to optimize it. A review of meta-modeling applications in structural optimization can be found in Barthelemy and Haftka (1993), and in multidisciplinary design optimization in Sobieszczanski-Sobieski and Haftka (1997).

As observed by many researchers, there is an important distinction between designs for computer experiments and designs for the more traditional response surface methods. Physical experiments exhibit random errors and computer experiments are often deterministic (cf. Simpson et al. 2004). This distinction is crucial and much research is therefore aimed at obtaining efficient designs for computer experiments.

As is recognized by several authors, such a design for computer experiments should at least satisfy the following two criteria (see Johnson et al. 1990, Morris and Mitchell 1995, and Simpson et al. 2001). First of all, the design should be *space-filling* in some sense. When no details on the functional behavior of the response parameters are available, it is important to be able to obtain information from the entire design space. Therefore, design points should be “evenly spread” over the entire region. One of the measures often used to obtain space-filling designs is the maximin measure (see p. 148 of Santner et al. 2003 and p. 17 of Forrester et al. 2006). The Audze-Eglais measure is another measure used for this purpose. Secondly, the design should be *non-collapsing*. When one of the design parameters has (almost) no influence on the function value, two design points that differ only in this parameter will “collapse”, i.e., they can be considered as the same point that is evaluated twice. For deterministic simulation models this is not a desirable situation. Therefore, two design points should not share any coordinate values when it is not known a priori which dimensions are important. Note that in other fields of research such designs are referred to as *low discrepancy* designs. To obtain non-collapsing designs the Latin hypercube structure is often enforced. It can be shown that if the function of interest is independent of one or more of the k parameters then, after removal of the irrelevant parameters, the projection of the LHD onto the reduced design space retains good spatial properties; see Koehler and Owen (1996). Maximin LHDs are frequently used in practical applications, see e.g., the examples given in Driessen et al. (2002), den Hertog and Stehouwer (2002), Alam et al. (2004), and Rikards and Auzins (2004).

Only a few authors consider the construction of maximin LHDs. For example, Morris and Mitchell (1995) used simulated annealing to find approximate maximin LHDs for up to five dimensions and up to 12 design points, and a few larger values, with respect to the ℓ^1 - and ℓ^2 -distance measure. van Dam et al. (2007) derived general formulas for two-dimensional maximin LHDs, when the distance measure is ℓ^∞ or ℓ^1 , while for the ℓ^2 -distance measure (approximate) maximin LHDs up to 1000 design points were obtained by using a branch-and-bound algorithm and constructing (adapted) periodic designs. Ye et al. (2000) proposed an exchange algorithm for finding approximate maximin symmetric LHDs. The symmetry property is used as a compromise between computing effort and design optimality. Jin et al. (2005) described an enhanced stochastic evolutionary (ESE) algorithm for finding approximate maximin LHDs. They also apply their method to other space-filling criteria. The Statistics Toolbox of Matlab also contains a function `lhsdesign` to generate approximate maximin LHDs. This function randomly generates a number of LHDs and picks the one with the largest separation distance. Although this method is very fast, other methods generally result in much better space-filling LHDs. To assess the quality of approximate maximin LHDs, van Dam et al. (2009) generated upper bounds on the separation distance for certain classes of maximin LHDs. By comparing the separation distances of LHDs to these bounds, we can get an indication of their quality.

There is much more literature related to maximin designs that are not restricted to LHDs. Note that a maximin design is certainly space-filling, but not necessarily non-collapsing.

First of all, the problem of finding the maximal common radius of n circles which can be packed into a square is equivalent to the maximin design problem in two dimensions. Melissen (1997) gives a comprehensive overview of the historical developments and state-of-the-art research in this field. For the ℓ^2 -distance measure in the two-dimensional case, optimal solutions are known for $n \leq 30$ and $n = 36$, see e.g., Kirchner and Wengerodt (1987), Peikert et al. (1991), Nurmela and Östergård (1999), and Markót and Csendes (2005). Furthermore, many good approximating solutions have been found for $n \geq 31$; see the Packomania website of Specht (2008). Baer (1992) solved the maximum ℓ^∞ -circle packing problem in a k -dimensional unit cube. The ℓ^1 -circle packing problem in a square has been solved for many values of n ; see Fejes Tóth (1971) and Florian (1989).

Secondly, the maximin design problem has been studied in location theory. In this area of research, the problem is usually referred to as the *max-min facility dispersion problem* (see Erkut 1990). Facilities are placed such that the minimal distance to any other facility is maximal. Again, the resulting solution is certainly space-filling, but not necessarily non-collapsing. A few papers consider maximin designs in higher dimensions, e.g., Trosset (1999), Locatelli and Raber (2002), Stinstra et al. (2003), and Dimnaku et al. (2005). These papers describe nonlinear programming heuristics to find approximate maximin designs. In most papers, a rectangular design space is assumed, but Trosset (1999), Stinstra et al. (2003) and Dimnaku et al. (2005) also specifically consider design spaces with different shapes.

Audze-Eglais LHDs are also constructed by only a few authors. The criterion was first introduced by Audze and Eglais (1977) and is based on the analogy of minimizing forces between charged particles. In Bates et al. (2004), the problem of finding

Audze-Eglais LHDs is formulated and a permutation genetic algorithm is used to generate them. Liefvendahl and Stocki (2006) compared maximin and Audze-Eglais LHDs and recommend the Audze-Eglais criterion over the maximin criterion. Examples of practical applications of Audze-Eglais LHDs can be found in Rikards et al. (2001), Bulik et al. (2004), Stocki (2005), and Hino et al. (2006).

There are several other measures proposed in the literature besides maximin and Audze-Eglais, e.g., maximum entropy, minimax, IMSE, and discrepancy. For a good overview, we refer to Koehler and Owen (1996). In statistical environments, Latin hypercube sampling (LHS) is often used. In such an approach, points on the grid are sampled without replacement, thereby deriving a random permutation for each dimension; see McKay et al. (1979). Giunta et al. (2003) give an overview of pseudo- and quasi-Monte Carlo sampling, LHS, orthogonal array sampling, and Hammersley sequence sampling. They notice that the basic LHS technique can lead to designs with poor space-filling properties. Extensions to the basic LHS technique are therefore necessary to obtain better designs but these are unfortunately not standard yet in all software packages. Bates et al. (1996) obtained designs for computer experiments by exploring so-called lattice points and using results from number theory.

Several papers combine space-filling criteria with the Latin hypercube structure. Jin et al. (2005) described an enhanced stochastic evolutionary algorithm for finding maximum entropy and uniform designs. van Dam (2008) derived interesting results for two-dimensional minimax LHDs. Rennen et al. (2010) consider nested maximin LHDs which consist of two separate designs, one being a subset of the other.

This paper is organized as follows. Section 2 describes how periodic designs can be used to obtain good approximate maximin and Audze-Eglais LHDs. In Sect. 3, we shortly describe some heuristics found in literature used for this same purpose. The ESE-algorithm of Jin et al. (2005) described in this section and periodic designs are used to generate new approximate maximin and Audze-Eglais LHDs. Computational results for up to ten dimensions and for up to 300 design points, as well as a comparison of the new and existing results, are provided in Sect. 4. Finally, Sect. 5 contains conclusions.

2 Periodic designs

Van Dam et al. (2007) showed that two-dimensional maximin Latin hypercube designs often have a nice, periodic structure. By constructing (adapted) periodic designs, many maximin LHDs and, otherwise, good LHDs, are found for up to 1000 points. Therefore, extending this idea to higher dimensions seems natural.

Let a k -dimensional Latin hypercube design of n points be represented by the sequences y_1, \dots, y_k , with every y_i a permutation of the set $\{0, \dots, n-1\}$. As in the two-dimensional case, a design is constructed by fixing the first dimension, without loss of generality, to the sequence $y_1 = (0, \dots, n-1)$ and assigning (adapted) periodic sequences to all other dimensions. Two types of periodic sequences are considered. The first one is the sequence (v_0, \dots, v_{n-1}) , where

$$v_i = (i+1)p \bmod (n+1) - 1, \quad \text{for } i = 0, \dots, n-1. \quad (3)$$

Here, p is the period of the sequence, which is chosen such that $n + 1$ and p have no common divisor, i.e., $\gcd(n + 1, p) = 1$, resulting in a permutation of the set $\{0, \dots, n - 1\}$.

Note that the periodic designs obtained in this way resemble *lattices*; see e.g., Bates et al. (1996). The main difference is that lattices are infinite sets of points, which may collapse, and, hence, to construct a (finite) Latin hypercube design a proper subset of non-collapsing lattice points should be chosen. For given n , the structure of the lattice will, however, not always lead to a Latin hypercube design with a sufficient number of points. This is in contrast to periodic designs, for which the modulo-operator insures that for every combination of periods p_j , with $\gcd(n + 1, p_j) = 1$, $j = 2, \dots, k$, a feasible Latin hypercube design is obtained.

The second type of sequence that is considered is the more general sequence (w_0, \dots, w_{n-1}) , where $w_i = (s + ip) \bmod n$ (note that we changed the modulus), for $i = 0, \dots, n - 1$. In this case, all starting points $s = 0, \dots, p$ and all periods $p = 1, \dots, \lfloor \frac{n}{2} \rfloor$ will be considered. Note, however, that the resulting sequence w may no longer be one-to-one, i.e., some values may occur more than once, and, hence, the resulting design may no longer be an LHD. Now, let $r > 0$ be the smallest value for which $w_r = w_0$; it then follows that $r = \frac{n}{\gcd(n, p)}$. When $r < n$ a way to construct a one-to-one sequence of length n is by shifting parts of the sequence by, say, q , and repeating this when necessary. To formulate this more explicitly, for the updated sequence w it now holds that

$$w_i = (s + ip + jq) \bmod n, \\ \text{for } i = jr, \dots, (j + 1)r - 1, \quad \text{and} \quad j = 0, \dots, \gcd(n, p) - 1. \quad (4)$$

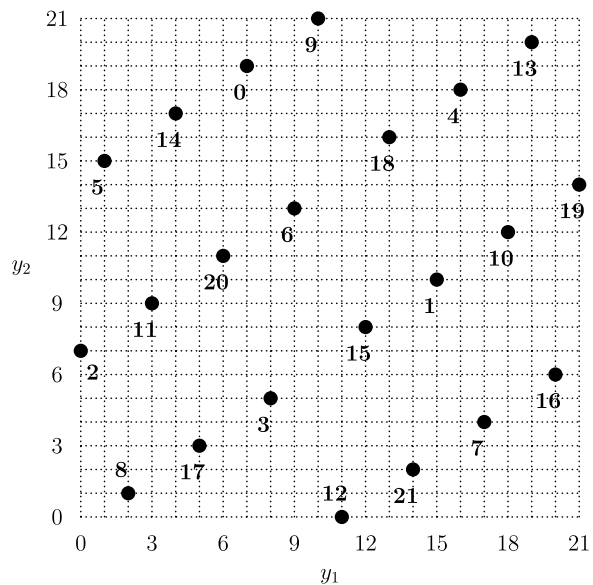
Let m represent the modulus and, hence, the type of sequence used, i.e., $m = n + 1$ corresponds to the first type and $m = n$ to the second. For given n , we now have to set the parameters (p, q, s, m) for every sequence y_2, \dots, y_k .

To find the best settings for the parameters it would be best to test all values. However, when the dimension and the number of points increase the number of possibilities increases rapidly. Hence, computing all possibilities gets very time-consuming or even impossible. Therefore, three classes of parameter settings (named A, B, and C) are distinguished. The largest one, class A, consists of checking the following parameter values: $p = 1, \dots, \lfloor \frac{n}{2} \rfloor$, $q = 1 - p, \dots, p - 1$, $s = 0, \dots, p$, and $m \in \{n, n + 1\}$. Testing in three and four dimensions indicated that almost all adapted periodic maximin designs are based on a shift of $1 - p$, -1 , or 1 (as was the case for two dimensions; see van Dam et al. (2007)). Furthermore, most maximin designs are found to have a starting point equal to either $p - 1$ or p . Class B is therefore set up to be a subset of class A with the aforementioned restrictions on the parameters q and s . Finally, for the dimensions 5 to 7 the number of possibilities has to be reduced even further, leading to parameter class C, which (based on some more test results) restricts class B to the values $q = 1$ and $s = p$, leaving the other parameters unchanged. Table 1 shows the different classes used in the computations of the approximate maximin LHDs for each dimension. For the approximate Audze-Eglais LHDs only class C is used.

As an example, consider a three-dimensional adapted periodic LHD of 22 points. For the maximin criterion, a best parameter setting (class A) is found to

Table 1 Different classes of periodic sequences are checked to generate maximin designs for each dimension

Dimension	Class A	Class B	Class C
3	$2 \leq n \leq 70$	$71 \leq n \leq 100$	–
4	$2 \leq n \leq 25$	$26 \leq n \leq 100$	–
5	–	$2 \leq n \leq 80$	$81 \leq n \leq 100$
6	–	$2 \leq n \leq 35$	$36 \leq n \leq 100$
7	–	–	$2 \leq n \leq 100$

Fig. 1 Two-dimensional projection of the three-dimensional LHD (y_1, y_2, y_3) of 22 points

be $(p_2, q_2, s_2, m_2) = (8, -7, 7, 22)$ and $(p_3, q_3, s_3, m_3) = (3, 0, 2, 23)$ and, hence, the corresponding maximin LHD, with separation distance 69, is defined by the sequences

$$\begin{aligned}
 y_1 &= (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21), \\
 y_2 &= (7, 15, 1, 9, 17, 3, 11, 19, 5, 13, 21, 0, 8, 16, 2, 10, 18, 4, 12, 20, 6, 14), \\
 y_3 &= (2, 5, 8, 11, 14, 17, 20, 0, 3, 6, 9, 12, 15, 18, 21, 1, 4, 7, 10, 13, 16, 19).
 \end{aligned}
 \tag{5}$$

Thus, y_3 is a periodic sequence, with $m = n + 1$, and y_2 is an adapted periodic sequence, with $m = n$ and $q_2 = -7$. Note that to obtain a one-to-one sequence, the second part of y_2 , i.e., $(0, 8, \dots, 14)$, is formed by shifting the first part of y_2 , i.e., $(7, 15, \dots, 21)$, by -7 . The periods and shift are clearly visible in the two-dimensional projection of the LHD in Fig. 1. In this figure the y_3 -values are depicted at the design points.

Like in the two-dimensional case, it may happen that for a given n the corresponding maximin LHD has a separation distance that is smaller than the distance of a design of $n - 1$ points. For these n , however, better designs can usually be derived

by adding an extra “corner point” to the LHD of $n - 1$ points. In this way, a monotone nondecreasing sequence of separation distances was found for all dimensions.

3 Other methods

3.1 Enhanced stochastic evolutionary algorithm

Besides restricting ourselves to a certain class of LHDs, we can also generate good maximin or Audze-Eglais LHDs using heuristics. The ESE-algorithm of Jin et al. (2005) is one of the methods developed for this purpose and is used in this paper to generate new approximate maximin and Audze-Eglais LHDs.

This method starts with an initial design and tries to find better designs by iteratively changing the current design. To determine if a new design is accepted, a threshold-based acceptance criterion is used. This criterion is controlled in the outer loop of the algorithm. In the inner loop of the algorithm new designs are explored.

The inner loop explores the design space as follows. At each iteration, the algorithm creates a fixed number of new designs by exchanging two randomly chosen elements. The new design with the largest separation distance or with the smallest Audze-Eglais objective value is then compared to the current design with a threshold criterion. The criterion is such that it ensures that better designs are always accepted and that worse designs can also be accepted with a certain probability. If the new design is accepted, it replaces the current design. This process is repeated for a user defined number of iterations.

The outer loop controls the threshold value. After the inner loop is completed, the outer loop determines how much improvement is made in the inner loop. If the amount of improvement is above a certain level, the algorithm starts an improving process in which it tries to rapidly find a local optimum. It does this by lowering the threshold value and thus accepting less deteriorations in the inner loop. If too little improvement is made, an exploration process is started which is intended to escape from a local optimum. The threshold value is first rapidly increased to move away from a local optimum and later slowly decreased to find better designs after moving away. The final design of the algorithm is the best design found during all iterations of the inner loop.

For a more detailed description of the algorithm, we refer to the original paper of Jin et al. (2005). To find maximin and Audze-Eglais LHDs, we implemented the ESE-algorithm in Matlab. The parameters of the algorithm were set to the values suggested in Jin et al. (2005). The only adjustment we made to the original algorithm is in the choice of stopping criterion. Instead of stopping after a fixed number of runs of the outer loop, our criterion is to stop when in the last 1000 runs of the outer loop no improvement is made.

3.2 Simulated annealing

Another heuristic used to find maximin LHDs is simulated annealing. Morris and Mitchell (1995) were the first to apply simulated annealing for this purpose. The simulated annealing method tries to find good designs by iteratively changing a random

starting design. These changes are chosen randomly from a predefined class of possible changes. For each design, these possible changes define a set of designs which are called the neighborhood of the design. Before a change is accepted, the new neighbor design obtained by applying the selected change is evaluated. If a change improves the current design, the change is always accepted. A key characteristic of simulated annealing is however that changes which result in a worse design can also be accepted. This enables simulated annealing to escape from local optima. A worse design is accepted with a probability which depends on two factors. Firstly, designs which are only slightly worse are accepted with a higher probability than design which are much worse. Secondly, the acceptance probability is changed during the course of the algorithm. Generally, worse designs are accepted with a higher probability at the beginning of the algorithm than at the end.

Besides Morris and Mitchell (1995), also Husslage (2006) used simulated annealing for finding maximin LHDs. One of the main differences between the two methods is the used objective function. Husslage (2006) directly used the separation distance of a design, whereas Morris and Mitchell (1995) used a surrogate measure ϕ_p . This measure also takes into account the number of pairs of points with a certain distance between them. By including this information, it is easier to decide which design is best if they have the same separation distance. This surrogate measure is also used by other authors like Jin et al. (2005) and Palmer and Tsui (2001).

Simulated annealing and ESE are similar in many respects. Both algorithms create new designs by changing a current design. Furthermore, both algorithms accept worse designs with a positive probability. The change of this acceptance probability in simulated annealing is similar to the change of the threshold value in the outer loop of ESE. The main difference between the two methods is that the ESE-algorithm creates several new designs and compares the best of these designs to the current design, whereas simulated annealing only creates one new design. The ESE-algorithm can thus be regarded as an enhancement of simulated annealing.

3.3 Permutation genetic algorithm

To obtain Audze-Eglais LHDs, Bates et al. (2004) used a permutation genetic algorithm. The genetic algorithm starts with generating a set of LHDs called a “population”. The Audze-Eglais distance of each design in this population is then calculated. Based on these distances, a subset of designs is selected using so-called elitist and tournament selection. A new population of designs is created by applying mutation and crossover operations to the selected designs. By repeatedly selecting and creating designs, the Audze-Eglais distances of the LHDs in the population gradually increase. Results of this algorithm were reported by Bates et al. (2004) for eight different combinations of n and k . In Sect. 4, we make a comparison between these results, the designs obtained with periodic designs, and the designs obtained with ESE.

4 Computational results

Using (adapted) periodic designs and the ESE-algorithm, approximate maximin and Audze-Eglais LHDs have been obtained for the cases described in Table 2. All computations have been performed on PCs with a 2.8-GHz Pentium D processor. For the cases with $n > 100$, a limit of 6 hours was imposed on the calculation time.

Table 2 Largest values of n for which LHDs were generated using periodic designs (PD) and using the ESE-algorithm

Dimension	3	4	5	6	7	8	9	10
Maximin PD	300	300	100	100	100			
Maximin ESE	300	300	100	100	100	100	100	100
Audze-Eglaiss PD	100	100	100					
Audze-Eglaiss ESE	100	100	100	100	100	100	100	100

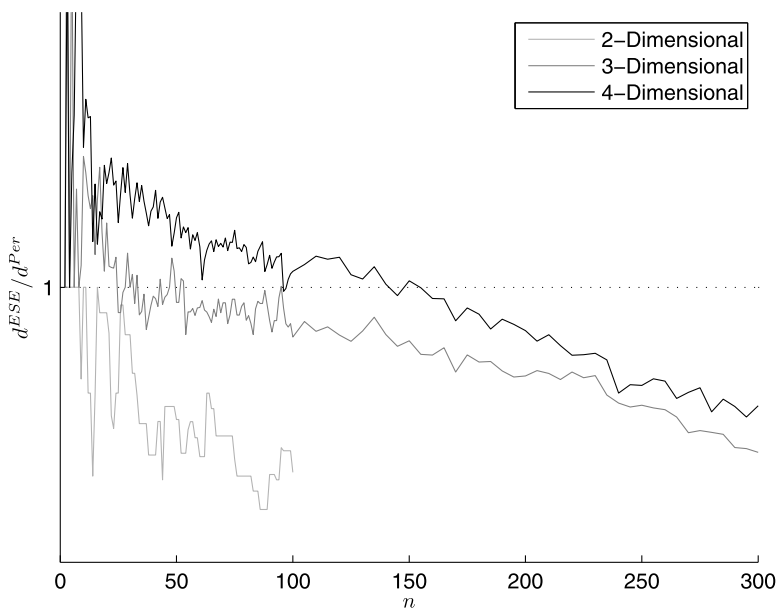
**Fig. 2** Ratio between separation distance of ESE and periodic designs

Table 5 shows the squared ℓ^2 -separation distance of the (approximate) maximin LHDs that were obtained by applying periodic designs and of those obtained by the ESE-algorithm. The column for two-dimensional periodic designs contains the results obtained in van Dam et al. (2007). From this table it can be seen that (adapted) periodic designs work particularly well for larger values of n and small k .

For dimension 2 to 4, Fig. 2 shows the ratio between the squared separation distance of ESE and periodic designs. A ratio larger than one indicates that the ESE design is better than the (adapted) periodic design. A break-even point, i.e., a point (or, better, an interval) where the preference shifts from the designs found by ESE to (adapted) periodic designs, is clearly visible in this figure. Furthermore, these break-even points seem to increase with the dimension of the design and it is to be expected that break-even points for k -dimensional designs, with $k \geq 5$, will occur for larger values of n , i.e., $n > 250$. Because all six- and seven-dimensional (adapted) periodic designs, of 3 to 100 points, are dominated by the designs found by ESE, the former are not computed for larger dimensions.

Table 3 Squared ℓ^2 -separation distance of designs found by Morris and Mitchell (1995) vs. the ESE-algorithm

n	3 dim		4 dim		5 dim		6 dim		7 dim		8 dim		9 dim	
	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE	M&M	ESE
3	6	6	7	7	8	8								
4	6	6	12	12	14	14								
5	11	11	15	15	24	24								
6	14	14	22	22	32	32	40	40						
7	17	17	28	28	40	40			61	61				
8	21	21	42	42	50	50					91	89		
9	22	22	42	42	61	61							126	126
10	27	27	50	47	82	82								
11	29	30	55	55	80	80								
12	36	36	63	63	91	91	139	136						
13														
14									219	215				

In Table 3, we compare the LHDs found by Morris and Mitchell (1995) and the ESE-algorithm. The ESE-algorithm is able to match the results of Morris and Mitchell (1995) for most combinations of k and n . Only for the cases $(k, n) = (4, 10)$, $(6, 12)$, $(7, 14)$, and $(8, 8)$ are slightly worse designs obtained. Three of these four design satisfy the property that $n = k$ or $n = 2k$. According to Morris and Mitchell (1995), these designs exhibit special symmetric properties; they refer to them as *foldover designs*. These special properties are probably the main explanation for the better results in these cases. For the case $(k, n) = (3, 11)$, we obtained an improved (and optimal) design. Furthermore, using a branch-and-bound algorithm, the three-dimensional designs of up to 15 points have been verified to be optimal (van Dam et al. 2009). From the above results, we can conclude that performances of the ESE-algorithm and the simulated annealing algorithm of Morris and Mitchell (1995) are closely matched. However, the numerical results of Morris and Mitchell (1995) are probably too limited to be useful in most practical applications.

We also compared the ESE results with the SA results in Husslage (2006) and saw that the ESE-algorithm gives better or equally good results for most combination of k and n . For only nine combinations the results are better of the SA algorithm and for 7 percent of the compared combinations the results are equally good. However, especially for larger values of n , the ESE algorithm found many designs with a more than 15 percent higher separation distance.

The results obtained for the Audze-Eglais measure are given in Table 6. We can easily see that the results of the ESE-algorithm are better for almost all cases. It is likely that by running ESE for some more starting solutions, better or equally good designs can be found for all cases. The ESE algorithm thus outperforms the periodic designs for the Audze-Eglais measure.

When we compare the results with those found by Bates in Table 4, we see that the ESE-algorithm gives better or equally good results. This shows that the ESE-algorithm is quite successful in finding LHDs with a good Audze-Eglais value.

Table 4 Audze-Eglais values of designs found by Bates et al. (2004) vs. the ESE-algorithm

n	2 dim		3 dim		5 dim	
	PermGA	ESE	PermGA	ESE	PermGA	ESE
5	1.2982	1.2982	0.7267	0.7267		
10	2.0662	2.0662	1.0242	1.0199		
50					0.7270	0.7195
120	5.5174	5.4941	1.9613	1.9328	0.7930	0.7840

5 Conclusions

This paper discusses existing and new results in the field of maximin and Audze-Eglais Latin hypercube designs. Such designs play an important role in the area of computer simulation. The new results are obtained using two heuristics. The first heuristic is based on the observation that many optimal LHDs, and two-dimensional LHDs in particular, exhibit a periodic structure. By considering periodic and adapted periodic designs, approximate maximin LHDs for up to seven dimensions and for up to 300 design points are constructed. The second heuristic uses the ESE-algorithm of Jin et al. (2005) to find approximate maximin LHDs for up to ten dimensions. These new results are compared to each other and to existing results obtained with simulated annealing and permutation genetic algorithms. In most cases, the ESE-algorithm resulted in the best maximin and Audze-Eglais LHDs. However when the number of design points is large with respect to the dimension of the design, the periodic designs tend to work better. Appendix gives the squared ℓ^2 -separation distances and Audze-Eglais values of the designs described in this paper. All corresponding designs can be downloaded from the website <http://www.spacefillingdesigns.nl>.

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Appendix: Tables of numerical results

Table 5 Squared ℓ^2 -separation distance found using periodic designs (PD) vs. the ESE-algorithm (ESE)

n	2 dim		3 dim		4 dim		5 dim		6 dim		7 dim		8 dim		9 dim		10 dim	
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per
2	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10
3	2	2	6	3	7	4	8	5	12	6	13	7	14	10	18	11	19	12
4	5	5	6	6	12	12	14	11	20	15	21	16	26	16	28	17	33	18
5	5	5	11	6	15	12	24	11	27	15	32	16	40	17	43	18	50	19
6	5	5	14	14	22	16	32	23	40	28	47	29	53	20	61	21	68	22
7	8	8	17	14	28	16	40	23	52	28	61	31	70	21	80	22	89	23

Table 5 (Continued)

<i>n</i>	2 dim		3 dim		4 dim		5 dim		6 dim		7 dim		8 dim		9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE	ESE
8	8	8	21	21	42	25	50	32	63	42	79	46	90	101	114	
9	8	10	22	21	42	25	61	39	75	45	92	47	112	126	142	
10	10	10	27	21	47	36	82	55	91	62	109	68	131	154	171	
11	10	10	30	24	55	39	80	55	108	62	129	69	152	178	206	
12	10	13	36	30	63	46	91	62	136	91	152	95	177	204	235	
13	10	13	41	35	70	51	103	64	138	91	178	95	205	235	268	
14	10	17	42	35	77	70	114	86	154	104	215	119	236	268	305	
15	13	17	45	42	87	71	129	88	171	111	220	129	273	309	347	
16	17	17	50	42	93	85	151	101	190	130	241	155	317	352	393	
17	17	18	53	42	99	85	158	113	208	131	266	161	332	396	442	
18	17	18	56	50	108	94	170	123	231	155	291	186	361	451	496	
19	17	18	59	57	119	94	184	136	256	169	323	195	390	469	554	
20	17	18	65	57	130	106	206	139	279	210	349	226	425	506	625	
21	18	20	68	65	145	116	223	165	302	210	380	236	463	548	650	
22	18	25	72	69	150	117	235	174	325	223	418	270	501	595	691	
23	18	26	75	72	159	130	250	178	348	236	448	273	542	640	747	
24	20	26	81	76	170	138	266	201	374	258	481	308	585	690	800	
25	20	26	86	91	178	156	285	205	400	286	520	350	626	739	857	
26	25	26	86	91	188	156	302	226	426	296	548	365	664	791	910	
27	25	26	90	91	198	157	310	238	447	310	585	382	712	840	976	
28	26	29	94	94	210	174	331	258	479	339	620	406	766	898	1041	
29	26	29	101	94	221	174	349	269	507	346	654	417	817	956	1100	
30	26	29	105	105	233	194	367	310	531	390	691	458	849	1019	1173	
31	26	32	110	107	244	212	405	310	563	390	728	482	900	1104	1241	
32	26	32	110	114	253	212	413	341	587	419	778	518	966	1139	1318	
33	26	34	117	114	264	215	426	341	622	430	814	537	1010	1201	1396	
34	26	37	125	133	273	230	445	358	648	470	851	561	1072	1270	1478	
35	26	37	126	133	286	234	467	366	683	495	914	586	1113	1326	1555	
36	26	37	131	133	297	250	486	400	719	518	939	636	1181	1405	1647	
37	26	37	138	152	309	266	520	408	744	528	976	668	1236	1477	1721	
38	26	41	142	152	321	283	541	415	788	561	1028	709	1286	1534	1790	
39	26	41	146	152	330	283	566	439	816	561	1084	726	1344	1609	1870	
40	26	41	152	155	342	291	575	492	876	632	1122	786	1416	1675	1946	
41	26	41	158	162	355	293	596	492	882	632	1156	802	1496	1765	2058	
42	29	41	161	168	367	319	626	496	907	670	1209	903	1526	1843	2149	
43	29	41	171	168	383	323	666	520	947	670	1256	903	1597	1905	2224	
44	29	50	179	186	396	331	680	548	992	696	1336	903	1653	1994	2319	
45	37	50	182	186	407	347	698	565	996	737	1366	926	1723	2079	2415	
46	37	50	186	189	421	366	723	592	1064	797	1408	985	1794	2155	2507	
47	37	50	189	189	438	378	754	611	1088	797	1459	985	1847	2244	2600	
48	37	50	201	189	450	413	763	632	1119	857	1531	1054	1924	2336	2732	

Table 5 (Continued)

<i>n</i>	2 dim		3 dim		4 dim		5 dim		6 dim		7 dim		8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE
49	37	50	203	196	464	415	803	634	1167	893	1592	1074	1989	2397	2828
50	37	52	206	213	478	415	830	663	1203	893	1639	1113	2041	2492	2893
51	37	52	206	213	490	421	850	692	1230	917	1662	1161	2132	2566	3006
52	37	58	217	213	504	455	883	709	1274	1003	1734	1231	2203	2686	3134
53	37	58	219	216	515	455	894	716	1340	1003	1808	1241	2234	2713	3261
54	37	58	209	233	534	477	932	760	1359	1019	1856	1288	2356	2805	3339
55	40	58	230	243	546	483	956	760	1421	1082	1896	1325	2429	2935	3452
56	41	58	230	243	558	515	982	784	1431	1104	2003	1358	2444	3021	3551
57	41	58	249	261	574	515	1007	846	1488	1136	2024	1479	2554	3119	3651
58	41	61	245	261	594	539	1035	846	1554	1166	2043	1479	2650	3187	3795
59	41	61	254	266	609	544	1063	849	1564	1223	2136	1509	2733	3297	3889
60	41	65	261	273	618	568	1094	904	1631	1242	2232	1577	2796	3420	4090
61	41	65	266	274	630	620	1128	904	1667	1258	2266	1615	2868	3525	4158
62	41	65	269	283	657	620	1150	934	1715	1306	2345	1680	2977	3636	4313
63	50	65	281	297	670	620	1178	967	1781	1380	2376	1680	3056	3690	4355
64	50	65	278	297	684	625	1206	985	1804	1430	2452	1769	3097	3820	4514
65	50	68	290	314	694	630	1216	997	1868	1430	2492	1786	3219	3932	4581
66	50	68	299	314	718	666	1261	1050	1874	1476	2543	1857	3279	4004	4769
67	50	74	294	314	735	666	1299	1072	1954	1482	2638	1868	3399	4081	4942
68	50	74	306	314	746	685	1330	1087	1983	1538	2693	1940	3453	4212	4995
69	50	74	306	324	765	698	1351	1112	2028	1588	2746	1965	3520	4317	5127
70	50	74	314	325	779	716	1378	1150	2094	1633	2838	2130	3588	4464	5276
71	50	74	314	325	793	716	1413	1150	2141	1644	2871	2130	3749	4548	5437
72	50	74	314	341	810	750	1430	1203	2136	1768	2960	2177	3810	4666	5556
73	50	74	329	350	834	759	1462	1229	2197	1768	3042	2206	3932	4776	5661
74	50	74	341	350	842	767	1512	1229	2291	1774	3120	2244	3941	4915	5817
75	50	80	341	350	867	771	1530	1274	2303	1862	3157	2295	4073	5006	5937
76	50	85	341	363	882	813	1569	1300	2387	1935	3218	2375	4178	5179	6111
77	50	85	341	363	894	823	1591	1308	2433	1947	3323	2403	4266	5222	6272
78	50	85	371	387	910	844	1621	1382	2479	2014	3387	2505	4390	5385	6384
79	50	85	374	387	927	848	1639	1382	2498	2037	3474	2525	4465	5535	6466
80	50	85	374	403	949	873	1691	1395	2554	2037	3550	2590	4565	5577	6653
81	50	85	381	406	963	916	1730	1406	2648	2064	3619	2642	4679	5748	6780
82	50	85	374	406	989	938	1742	1475	2680	2141	3669	2753	4719	5859	6935
83	50	90	374	417	1002	940	1762	1501	2696	2141	3723	2767	4848	5976	7094
84	50	90	406	426	1021	967	1818	1534	2790	2229	3870	2838	4920	6119	7256
85	50	90	413	426	1043	967	1866	1552	2819	2232	3919	2874	5032	6212	7357
86	50	97	413	428	1053	967	1882	1573	2875	2375	3958	3103	5164	6346	7532
87	50	97	413	428	1073	976	1934	1598	2913	2375	4095	3103	5225	6469	7639
88	50	97	434	437	1086	1050	1954	1685	2975	2398	4166	3183	5340	6660	7877
89	50	97	426	443	1102	1050	1990	1690	3067	2400	4176	3183	5450	6750	7950

Table 5 (Continued)

<i>n</i>	2 dim		3 dim		4 dim		5 dim		6 dim		7 dim		8 dim		9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE	ESE
90	58	98	446	481	1134	1060	2027	1710	3104	2516	4308	3190	5576	6901	8128	
91	58	98	434	481	1134	1089	2031	1748	3143	2516	4379	3234	5626	6950	8330	
92	58	98	446	481	1149	1089	2100	1805	3216	2599	4428	3277	5758	7067	8442	
93	58	100	446	481	1171	1098	2130	1813	3283	2604	4512	3361	5832	7342	8601	
94	58	100	470	481	1199	1124	2169	1881	3348	2747	4581	3474	6007	7436	8774	
95	65	100	482	481	1219	1135	2206	1901	3335	2747	4703	3531	6064	7469	8877	
96	65	101	486	509	1250	1261	2227	1965	3451	2769	4808	3639	6222	7645	9146	
97	65	101	474	515	1258	1261	2299	1965	3514	2817	4848	3639	6304	7781	9379	
98	65	101	485	531	1283	1261	2299	1965	3560	2850	4936	3690	6376	7896	9381	
99	65	101	489	531	1298	1261	2338	2009	3628	2878	4999	3731	6448	8023	9617	
100	65	109	494	554	1305	1261	2401	2053	3648	3000	5040	3903	6617	8228	9835	
105			521	563	1395	1329										
110			566	626	1510	1414										
115			594	650	1591	1499										
120			629	702	1708	1603										
125			629	713	1798	1750										
130			693	766	1906	1872										
135			729	780	1995	1909										
140			758	845	2103	2089										
145			779	894	2185	2225										
150			825	934	2310	2278										
155			842	986	2365	2367										
160			854	1002	2486	2548										
165			904	1041	2582	2648										
170			914	1121	2659	2869										
175			965	1132	2771	2902										
180			1011	1208	2897	3077										
185			1026	1224	2970	3267										
190			1061	1298	3094	3325										
195			1086	1350	3210	3492										
200			1106	1371	3257	3596										
205			1166	1425	3273	3708										
210			1196	1473	3377	3767										
215			1229	1538	3476	3983										
220			1259	1544	3543	4159										
225			1293	1611	3661	4292										
230			1329	1646	3703	4326										
235			1305	1706	3815	4532										
240			1350	1806	3893	5061										
245			1397	1891	3986	5061										
250			1412	1901	3990	5075										

Table 5 (Continued)

n	2 dim		3 dim		4 dim		5 dim		6 dim		7 dim		8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE
255			1417	1923	4100	5122									
260			1445	1971	4164	5236									
265			1449	2021	4182	5519									
270			1464	2144	4361	5656									
275			1478	2150	4487	5746									
280			1493	2184	4388	6023									
285			1501	2209	4607	6094									
290			1476	2269	4722	6380									
295			1526	2354	4726	6590									
300			1542	2409	4898	6604									

Table 6 Audze-Eglais values found using periodic designs (PD) vs. the ESE-algorithm (ESE)

n	2 dim		3 dim		4 dim		5 dim		6 dim	7 dim	8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE	ESE	ESE
2	0.500	0.500	0.333	0.333	0.250	0.250	0.200	0.200	0.167	0.143	0.125	0.111	0.100
3	0.900	0.900	0.611	0.611	0.386	0.450	0.321	0.362	0.250	0.230	0.193	0.200	0.151
4	1.000	1.000	0.642	0.642	0.454	0.489	0.367	0.382	0.300	0.260	0.225	0.201	0.180
5	1.298	1.390	0.727	0.891	0.509	0.658	0.401	0.527	0.336	0.287	0.250	0.222	0.200
6	1.521	1.521	0.794	0.800	0.561	0.594	0.431	0.476	0.358	0.307	0.268	0.238	0.215
7	1.598	1.598	0.867	0.975	0.599	0.694	0.464	0.532	0.376	0.322	0.282	0.250	0.225
8	1.804	1.879	0.921	0.960	0.619	0.696	0.488	0.538	0.398	0.334	0.292	0.260	0.234
9	1.935	1.935	0.971	1.052	0.660	0.742	0.504	0.567	0.414	0.349	0.301	0.267	0.240
10	2.066	2.066	1.020	1.085	0.686	0.744	0.515	0.556	0.425	0.360	0.311	0.273	0.246
11	2.196	2.279	1.069	1.137	0.709	0.785	0.536	0.612	0.434	0.369	0.319	0.281	0.250
12	2.273	2.273	1.095	1.163	0.724	0.785	0.551	0.589	0.441	0.375	0.326	0.287	0.256
13	2.401	2.487	1.128	1.191	0.746	0.825	0.563	0.632	0.453	0.381	0.331	0.292	0.261
14	2.476	2.476	1.167	1.252	0.762	0.829	0.575	0.635	0.462	0.385	0.335	0.296	0.265
15	2.578	2.643	1.194	1.255	0.775	0.818	0.583	0.636	0.470	0.393	0.339	0.299	0.268
16	2.666	2.683	1.221	1.290	0.791	0.848	0.589	0.642	0.477	0.398	0.341	0.302	0.271
17	2.721	2.721	1.246	1.340	0.805	0.866	0.600	0.656	0.483	0.404	0.347	0.305	0.273
18	2.819	2.848	1.271	1.337	0.816	0.875	0.609	0.655	0.488	0.408	0.350	0.307	0.275
19	2.890	2.984	1.292	1.374	0.827	0.895	0.615	0.667	0.492	0.413	0.354	0.310	0.277
20	2.959	2.962	1.318	1.394	0.835	0.907	0.620	0.681	0.496	0.416	0.358	0.313	0.278
21	3.025	3.033	1.339	1.408	0.847	0.914	0.625	0.671	0.501	0.419	0.361	0.316	0.281
22	3.070	3.070	1.357	1.426	0.856	0.922	0.632	0.687	0.505	0.422	0.363	0.318	0.283
23	3.138	3.159	1.377	1.454	0.868	0.925	0.638	0.693	0.510	0.425	0.366	0.321	0.285
24	3.197	3.201	1.396	1.458	0.875	0.931	0.644	0.677	0.513	0.427	0.368	0.323	0.287
25	3.254	3.293	1.412	1.485	0.884	0.940	0.648	0.701	0.516	0.430	0.370	0.324	0.289

Table 6 (Continued)

<i>n</i>	2 dim		3 dim		4 dim		5 dim		6 dim		7 dim	8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE	ESE	ESE	ESE
26	3.309	3.332	1.428	1.480	0.891	0.947	0.653	0.707	0.518	0.432	0.372	0.326	0.290	
27	3.360	3.383	1.442	1.499	0.898	0.957	0.657	0.708	0.521	0.435	0.373	0.328	0.292	
28	3.405	3.420	1.454	1.503	0.906	0.961	0.660	0.712	0.524	0.437	0.375	0.329	0.293	
29	3.458	3.539	1.468	1.543	0.912	0.978	0.664	0.716	0.527	0.439	0.376	0.330	0.294	
30	3.505	3.515	1.481	1.528	0.919	0.974	0.667	0.716	0.530	0.441	0.378	0.331	0.295	
31	3.543	3.550	1.493	1.563	0.925	0.976	0.671	0.719	0.533	0.443	0.380	0.333	0.296	
32	3.589	3.623	1.505	1.562	0.931	0.996	0.674	0.729	0.535	0.444	0.381	0.334	0.297	
33	3.636	3.642	1.517	1.588	0.935	0.990	0.678	0.732	0.537	0.446	0.383	0.335	0.298	
34	3.676	3.713	1.528	1.565	0.941	1.005	0.682	0.735	0.540	0.447	0.384	0.336	0.299	
35	3.716	3.786	1.539	1.601	0.946	1.003	0.685	0.731	0.542	0.449	0.385	0.337	0.300	
36	3.758	3.774	1.549	1.600	0.950	1.005	0.688	0.734	0.543	0.450	0.386	0.338	0.301	
37	3.794	3.819	1.558	1.599	0.956	1.019	0.691	0.736	0.545	0.452	0.387	0.339	0.301	
38	3.828	3.828	1.568	1.623	0.959	1.023	0.694	0.746	0.547	0.453	0.388	0.340	0.302	
39	3.868	3.879	1.578	1.646	0.965	1.025	0.696	0.742	0.548	0.455	0.389	0.341	0.303	
40	3.906	3.918	1.587	1.636	0.968	1.019	0.699	0.742	0.550	0.456	0.390	0.341	0.303	
41	3.939	4.009	1.596	1.639	0.971	1.033	0.701	0.751	0.551	0.457	0.391	0.342	0.304	
42	3.974	3.974	1.604	1.658	0.975	1.031	0.703	0.742	0.552	0.458	0.392	0.343	0.305	
43	4.007	4.045	1.612	1.675	0.979	1.042	0.705	0.752	0.554	0.460	0.393	0.344	0.306	
44	4.029	4.029	1.621	1.670	0.983	1.040	0.708	0.754	0.555	0.461	0.394	0.344	0.306	
45	4.063	4.074	1.628	1.678	0.986	1.044	0.710	0.752	0.557	0.462	0.394	0.345	0.307	
46	4.096	4.115	1.636	1.693	0.990	1.044	0.712	0.753	0.559	0.463	0.395	0.346	0.307	
47	4.130	4.179	1.643	1.695	0.993	1.055	0.714	0.761	0.560	0.464	0.396	0.346	0.308	
48	4.160	4.206	1.650	1.699	0.997	1.052	0.716	0.759	0.561	0.464	0.397	0.347	0.308	
49	4.187	4.187	1.657	1.711	1.001	1.059	0.718	0.762	0.563	0.465	0.398	0.347	0.309	
50	4.216	4.254	1.665	1.713	1.004	1.058	0.720	0.765	0.564	0.466	0.398	0.348	0.309	
51	4.246	4.280	1.671	1.729	1.007	1.063	0.721	0.768	0.566	0.467	0.399	0.348	0.310	
52	4.273	4.277	1.678	1.730	1.010	1.062	0.723	0.765	0.567	0.468	0.400	0.349	0.310	
53	4.302	4.343	1.685	1.734	1.013	1.072	0.725	0.771	0.568	0.468	0.400	0.349	0.310	
54	4.331	4.341	1.690	1.739	1.016	1.070	0.726	0.769	0.569	0.469	0.401	0.350	0.311	
55	4.355	4.413	1.697	1.755	1.018	1.073	0.728	0.773	0.570	0.470	0.401	0.350	0.311	
56	4.382	4.404	1.703	1.756	1.022	1.071	0.729	0.772	0.571	0.470	0.402	0.351	0.312	
57	4.404	4.427	1.708	1.760	1.024	1.079	0.731	0.776	0.572	0.471	0.403	0.351	0.312	
58	4.431	4.437	1.714	1.763	1.027	1.076	0.732	0.776	0.573	0.472	0.403	0.352	0.312	
59	4.458	4.498	1.719	1.777	1.030	1.087	0.734	0.780	0.574	0.473	0.404	0.352	0.313	
60	4.482	4.490	1.725	1.772	1.032	1.079	0.735	0.777	0.575	0.473	0.404	0.353	0.313	
61	4.499	4.530	1.731	1.778	1.034	1.087	0.736	0.776	0.576	0.474	0.405	0.353	0.314	
62	4.526	4.576	1.736	1.786	1.036	1.087	0.738	0.781	0.576	0.475	0.405	0.354	0.314	
63	4.556	4.576	1.742	1.789	1.039	1.094	0.739	0.783	0.577	0.475	0.406	0.354	0.314	
64	4.573	4.590	1.746	1.794	1.041	1.087	0.740	0.784	0.578	0.476	0.406	0.354	0.315	
65	4.595	4.599	1.751	1.802	1.043	1.095	0.742	0.786	0.579	0.477	0.407	0.355	0.315	
66	4.619	4.635	1.757	1.804	1.045	1.093	0.742	0.785	0.580	0.477	0.407	0.355	0.315	

Table 6 (Continued)

<i>n</i>	2 dim		3 dim		4 dim		5 dim		6 dim	7 dim	8 dim	9 dim	10 dim
	ESE	Per	ESE	Per	ESE	Per	ESE	Per	ESE	ESE	ESE	ESE	ESE
67	4.636	4.642	1.761	1.812	1.047	1.100	0.744	0.787	0.581	0.478	0.407	0.355	0.315
68	4.661	4.681	1.766	1.819	1.049	1.096	0.745	0.790	0.581	0.478	0.407	0.356	0.316
69	4.683	4.713	1.771	1.818	1.052	1.104	0.746	0.791	0.582	0.479	0.408	0.356	0.316
70	4.703	4.710	1.775	1.818	1.053	1.100	0.747	0.790	0.583	0.480	0.408	0.356	0.316
71	4.727	4.742	1.780	1.831	1.055	1.108	0.747	0.795	0.584	0.480	0.409	0.357	0.317
72	4.743	4.746	1.784	1.829	1.057	1.103	0.749	0.791	0.584	0.481	0.409	0.357	0.317
73	4.763	4.781	1.789	1.836	1.059	1.106	0.749	0.794	0.585	0.481	0.409	0.357	0.317
74	4.781	4.820	1.793	1.842	1.061	1.112	0.751	0.795	0.586	0.482	0.410	0.358	0.317
75	4.803	4.817	1.796	1.847	1.063	1.112	0.752	0.797	0.586	0.482	0.410	0.358	0.318
76	4.823	4.828	1.801	1.850	1.064	1.110	0.753	0.796	0.587	0.483	0.410	0.358	0.318
77	4.838	4.853	1.805	1.851	1.066	1.112	0.755	0.799	0.587	0.483	0.411	0.359	0.318
78	4.863	4.883	1.809	1.847	1.068	1.114	0.755	0.795	0.588	0.484	0.411	0.359	0.318
79	4.882	4.934	1.812	1.863	1.070	1.119	0.756	0.801	0.589	0.484	0.411	0.359	0.318
80	4.895	4.922	1.816	1.864	1.071	1.117	0.757	0.799	0.589	0.484	0.412	0.359	0.319
81	4.920	4.942	1.820	1.869	1.072	1.120	0.758	0.802	0.590	0.485	0.412	0.360	0.319
82	4.936	4.944	1.824	1.862	1.074	1.120	0.759	0.801	0.590	0.485	0.413	0.360	0.319
83	4.949	4.949	1.827	1.879	1.076	1.126	0.760	0.805	0.591	0.486	0.413	0.360	0.319
84	4.968	4.992	1.831	1.876	1.077	1.122	0.761	0.802	0.591	0.486	0.413	0.360	0.320
85	4.985	5.014	1.834	1.879	1.079	1.124	0.761	0.804	0.592	0.486	0.414	0.360	0.320
86	5.003	5.014	1.838	1.882	1.081	1.125	0.762	0.804	0.592	0.487	0.414	0.361	0.320
87	5.019	5.060	1.842	1.891	1.082	1.130	0.763	0.808	0.593	0.487	0.414	0.361	0.320
88	5.034	5.047	1.845	1.885	1.083	1.130	0.764	0.805	0.594	0.487	0.414	0.361	0.320
89	5.056	5.096	1.848	1.895	1.085	1.133	0.765	0.810	0.594	0.488	0.415	0.361	0.321
90	5.070	5.063	1.852	1.885	1.086	1.131	0.766	0.807	0.594	0.488	0.415	0.361	0.321
91	5.086	5.113	1.854	1.890	1.088	1.134	0.766	0.809	0.595	0.489	0.415	0.362	0.321
92	5.104	5.114	1.858	1.902	1.089	1.135	0.767	0.809	0.595	0.489	0.416	0.362	0.321
93	5.119	5.122	1.861	1.903	1.090	1.136	0.768	0.810	0.596	0.489	0.416	0.362	0.321
94	5.130	5.143	1.864	1.900	1.092	1.138	0.769	0.810	0.596	0.490	0.416	0.362	0.321
95	5.151	5.177	1.867	1.909	1.093	1.138	0.769	0.813	0.597	0.490	0.416	0.362	0.322
96	5.163	5.183	1.870	1.910	1.094	1.139	0.770	0.811	0.597	0.490	0.417	0.363	0.322
97	5.177	5.179	1.872	1.915	1.096	1.138	0.771	0.814	0.598	0.490	0.417	0.363	0.322
98	5.198	5.223	1.876	1.915	1.097	1.142	0.771	0.812	0.598	0.491	0.417	0.363	0.322
99	5.211	5.244	1.879	1.923	1.098	1.143	0.772	0.815	0.599	0.491	0.417	0.363	0.322
100	5.223	5.221	1.882	1.921	1.099	1.143	0.773	0.812	0.599	0.491	0.418	0.363	0.322

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